The behavior of $f(R)$ gravity in the solar system, galaxies and clusters

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Cosmologically interesting $f(R)$ gravity models are in general strongly environment dependent. For these models, we derive the complete sets of the linearized field equations in the Newtonian gauge, under environments of the solar system, galaxies and clusters respectively. In the solar system, the matter density is much higher than the cosmological critical density. This results in significant suppression on corrections to the general relativity (GR) induced by $f(R)$ gravity. We show that the behavior of $f(R)$ gravity in the solar system can be virtually identical to that of GR.

Although the environments in galaxies and clusters differ from that in the solar system, we find that gravitational lensing of galaxies and clusters are virtually identical to that in GR. Fortunately, galaxy rotation curve and intra-cluster gas pressure profile may contain valuable information to distinguish between $f(R)$ gravity and GR.

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INTRODUCTION

The standard theory of gravity (the general relativity (GR)) combined with the standard model of particle physics failed to explain a wide range of independent observations, from the expansion of the Universe, the cosmic microwave background, the large scale structure of the universe to galaxy and cluster dynamics. To reconcile observations, dark matter and dark energy, as modifications to particle physics, were proposed and work surprisingly well [1]. However, equally reasonable in logic, one can modify gravity instead to reconcile observations. It has been shown that the modified Newtonian dynamics (MOND) and its relativistic version Tensor-Vector-Scalar theory [2] can replace dark matter at galaxy scales to reproduce galaxy rotation curves, and 5-D DGP gravity [3] and $f(R)$ gravity [4] can replace dark energy to reproduce the accelerated expansion of the universe.

Like dark matter and dark energy, viable modifications in gravity must pass all sorts of tests from the large scale structure of the universe [5, 6, 7, 8] to galaxy and cluster dynamics to the solar system tests (SST). Unlike GR, which involves metric derivatives no higher than second order, $f(R)$ gravity involves also third and fourth order derivatives, which caused complications in the calculation [9]

An outstanding question is whether $f(R)$ gravity is consistent with SST, which have put stringent constraints on the PPN parameter $\gamma = 1 \pm O(10^{-4})$ [10]. Various authors have discussed the conditions for $f(R)$ gravity or its extentions to pass SST [11]. More specifically, [12, 13, 14] argued that $f(R)$ gravity can in general be perfectly consistent with SST, while [15, 16, 17] claimed that a wide range of $f(R)$ gravity models posses a $\gamma = 1/2$ and is thus ruled out.

To clarify this crucial issue, we derive the complete set of linearized field equations of the two Newtonian potentials ϕ and ψ, for cosmologically interesting $f(R)$ gravity models. The field equations turn out to take simple forms under the environments of the solar system, galaxies or clusters. For the idealized case of the Sun embedded in a uniform background with density $\bar{\rho}$, they can be solved analytically. We find that these equations accept the constant curvature (where $r \neq 0$) solution with $\gamma = 1$. Furthermore, we find that the large matter density in the solar system, when compared to the cosmological mean, significantly suppresses the corrections induced to GR by $f(R)$ gravity. The field equations thus are virtually identical to that in GR. So $f(R)$ gravity should pass all SST.

Furthermore, we perturb the FRW background and derive the equations applicable to galaxies and clusters. We find virtually identical gravitational lensing in both $f(R)$ gravity and GR. However, galaxy rotation curve, cluster pressure profile, the relation between the cluster mass, X-ray temperature, X-ray luminosity and the SZ flux, are modified in $f(R)$ gravity.

LINEARIZED FIELD EQUATIONS OF $f(R)$ GRAVITY

The $f(R)$ gravity takes the action

$$L = \int (R + f(R))\sqrt{-g} d^4x \ , \quad (1)$$

and the field equation

$$R_{uv} - \frac{1}{2} g_{uv}(R+f) + f_R R_{uv} + g_{uv}\Box f_R - f_{R;u;v} = 8\pi G T_{uv} \ . \quad (2)$$

Throughout the paper, we assume that $T_{\mu\nu}$ takes the form of ideal fluid with negligible pressure. For $f(R)$

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gravity models of cosmological interest, the $f(R)$ term generally vanishes in high density environment, in order not to conflict with early time physics such as BBN and CMB, and becomes comparable to R in environment where $\rho \sim \rho_c$, in order to drive late time acceleration. Here, ρ_c is the cosmological critical density. For a wide range of such models, $|f_R| \ll 1$ at $R \sim R_{\rm solar} \sim R_{\rm FRW}(z \gg 1)$. Here $R_{\rm solar}$ is some typical value of the curvature scalar in the solar system and $R_{\rm FRW}(z \gg 1)$ is the curvature scalar of the FRW background at early universe. Throughout this paper, we adopt these conditions.

In the solar system, galaxies and clusters, we expect that the gravitational field is weak and the time variation of the field is negligible. Due to different environments and different boundary conditions, the linearized field equations in the solar system differ slightly from that in galaxies and clusters. Thus we will treat the two cases separately.

Field equations applicable to the solar system

In the solar system, we choose a static metric with the proper time

$$\begin{aligned} ds^2 &= -g_{\mu\nu}dx^\mu dx^\nu \\ &= (1+2\psi(\mathbf{x}))dt^2 - (1+2\phi(\mathbf{x}))\sum_i dx^{i,2} \ . \end{aligned} \quad (3)$$

This is just the widely adopted Newtonian gauge in cosmology when dropping the time dependence, where ϕ and ψ are two Newtonian potentials. Since $|\phi|, |\psi| \ll 1$, non-vanishing Ricci tensor components are $R_{00} \simeq \nabla^2\psi$ and $R_{ij} \simeq -\nabla^2\phi\delta_{ij} - (\phi+\psi)_{,ij}$. The curvature scalar $R \simeq -2\nabla^2\psi - 4\nabla^2\phi$. In Eq. 2, it is safe to neglect terms $(R+f)\phi$, $(R+f)\psi$ with respect to $R+f$ and neglect terms $\phi\Box f_R$, $\psi\Box f_R$ with respect to $\Box f_R$, since ϕ, ψ are small. Also, one can approximate the covariant derivative $f_{R;i;j}$ as the ordinary derivative $f_{R,ij}$, since $|\Gamma^\sigma_{ij} f_{R;\sigma}/f_{R,ij}| \sim |\phi| \ll 1$. We then obtain

$$R_{00}(1+f_R) + \frac{1}{2}(R+f) - \Box f_R = 8\pi G\rho \ , \quad (4)$$

$$R_{ii}(1+f_R) - \frac{1}{2}(R+f) - \Box f_R - (f_R)_{,ii} = 0 \ , \quad (5)$$

$$R_{ij}(1+f_R) - (f_R)_{,ij} = 0 \ \ \text{when } i \neq j \ . \quad (6)$$

Before proceeding to the final results, we point out a generic constraint exerted by Eq. 6, for constant curvature solutions. For this kind of solutions, $f_{R;i;j} = f_{R,ij} = 0$ and $(\phi+\psi)_{,ij} = 0$. Thus the coefficient of the r^{-1} term in ϕ and ψ must be equal (with opposite sign). In another word, *the constant curvature solution must have* $\gamma = 1$.

Given the condition $|f_R| \ll 1$ or R is a constant, Eq. 6 can be integrated to give

$$(\phi+\psi)(1+f_R) = -f_R + C_0 r^2 + \text{const.} \ . \quad (7)$$

The integral would produce a term $ax_i + bx_j$ in the right hand side of Eq. 7. However there is no special direction in the Universe, so it vanishes. The term $C_0 r^2$ is necessary. It reflects the fact that, the flat Minkowski space-time is no longer the true background in $f(R)$ gravity. Eq. 7 holds beyond the condition $|f_R| \ll 1$ or R is a constant. [18]

Combining Eq. 4, 5 and 7, we obtain

$$\nabla^2(\phi-\psi) = -\frac{8\pi G\rho + 2C_0}{1+f_R} \ . \quad (8)$$

Eq. 7 and 8 completely determine the gravitational field of $f(R)$ gravity, up to a constant C_0. C_0 can be determined by either Eq. 4, 5, or the trace of Eq. 2: $(f_R-1)R - 2f + 3\Box f_R = -8\pi G\rho$. For example, when $f = \Lambda$ is the cosmological constant, one can show $C_0 = f/8$.

These equations do not require the condition of spherical symmetry and can be applied to various environments including star forming regions and star or black hole accretion disk. However, to clarify the issue whether $f(R)$ gravity is consistent with SST, we apply them to an idealized case, where a point source with mass M (the Sun) is embedded in a uniform background with density $\bar\rho$. We draw the attention that the vacuum ($\rho = 0$) solution investigated in the literature is not really relevant for SST, as also noticed by [8, 14]. The behavior of $f(R)$ gravity is highly environment dependent. The local density ρ where SST were carried out is much higher than ρ_c ($\rho/\rho_c \gtrsim 10^6$-10^8, [8]). Thus the behavior of $f(R)$ gravity in the solar system can differ dramatically from that in the vacuum.

In Eq. 8, we take the limit $|f_R| \ll 1$ and obtain

$$\phi - \psi = \frac{2GM}{r} - \frac{4\pi G\bar\rho + C_0}{3} r^2 + \text{const.} \quad (9)$$

Combining Eq. 7, we obtain

$$\begin{aligned} \phi &= \frac{GM}{r} + \left(-\frac{2\pi G\bar\rho}{3} + \frac{C_0}{3}\right) r^2 \ , \\ \psi &= -\frac{GM}{r} + \left(\frac{2\pi G\bar\rho}{3} + \frac{2C_0}{3}\right) r^2 \ , \end{aligned} \quad (10)$$

where the constant C_0 is given by the trace of Eq. 2

$$[f_R R - 2f]\,|_{R=8\pi G\bar\rho - 16C_0} + 16C_0 = 0 \quad (11)$$

Here, $\Box f_R = 0$ since R is a constant for such solution. One can express the solution in a more familiar form of the Schwarzschild metric, by variable transform $r \to r' = r(1+\phi)$. When $\bar\rho = 0$, the Schwarzschild-de Sitter space-time solution found by [12] is recovered.

Is this solution consistent with all approximations we made? Yes. The weak field condition is satisfied since $|\phi|, |\psi| \sim 10^{-8}[\mathrm{AU}/r] \ll 1$. The condition $|f_R| \ll 1$ is perfectly satisfied too . Since $\bar{\rho}/\rho_c \gtrsim 10^6$-$10^8$ [8], the solution of Eq. 11 is $R \simeq 8\pi G\bar{\rho}$ and $C_0 \simeq (2f - f_R R)|_{R=8\pi G\bar{\rho}}/16 \ll R$. For $f(R) = -\mu^4/R$ to drive the late time acceleration, $\mu \sim H_0$ where H_0 is the present day Hubble constant. We then have $|f_R| \sim (\bar{\rho}/\rho_c)^2 \lesssim 10^{-12}$-$10^{-14}$. For $f(R) = -\lambda_1 H_0^2 \exp(-R\lambda_2 H_0^2)$ proposed in [8], $\lambda_1 \sim 1$ and $\lambda_2 \sim 10^3$ can produce virtually degenerate expansion rate to that of ΛCDM cosmology. For these values, $|f_R| \sim 10^{-400}$.

Is it the only solution? In the limit $|f_R| \ll 1$, yes. In this limit, Eq. 7 gives $\nabla^2(\phi+\psi) = -\nabla^2 f_R + 6C_0$ and Eq. 8 gives $\nabla^2(\phi-\psi) = -8\pi G\bar{\rho} - 2C_0$. Since $R = -2\nabla^2\psi - 4\nabla^2\phi$, we have $\nabla^2 f_R = (R - 8\pi G\bar{\rho} + 16C_0)/3$. Plug it into the trace of Eq. 2, one finds $f_R R - 2f + 16C_0 = 0$. R, as solutions to this equation, then must be constant, having the same value given by Eq. 11.

Is it consistent with SST? Yes. Since $C_0 \ll G\rho$, Eq. 10 is virtually identical to that in GR and causes no conflict with SST. Furthermore, in the solar system, $|\phi|, |\psi| \sim 10^{-8}(M/M_{\mathrm{Sun}})(\mathrm{AU}/r) \gg |f_R|$ is generally satisfied. $|C_0 r^2/\phi| \ll 1$ is also satisfied. We then have $\phi+\psi \simeq 0$ and $\nabla^2(\phi-\psi) \simeq -8\pi G\rho$. Thus the field equations are virtually identical to that in GR and we should sense no difference between $f(R)$ gravity and GR.

Field equations applicable to galaxies and clusters

Galaxies and clusters are virialized objects embedded in the FRW background. Approximately they are static in the physical coordinate. So the time scale of the field variation is the Hubble time and it is safe to neglect all the time derivatives of ϕ and ψ. For a galaxy or cluster at $z = 1/a - 1$, we choose a metric

$$ds^2 = (1+2\psi)dt^2 - a^2(1+2\phi)\sum_i dx^{i,2} \quad . \qquad (12)$$

Eq. 7 is then replaced by [8]

$$\phi + \psi = -f_R(R_{\mathrm{FRW}} + \delta R) + f_R(R_{\mathrm{FRW}}) \; . \qquad (13)$$

Here, $R = R_{\mathrm{FRW}} + \delta R = 6(\dot{a}^2/a^2 + \ddot{a}/a) - 2\nabla^2\psi - 4\nabla^2\phi$. Throughout this section, the derivative is with respect to the physical coordinate. The term $C_0 r^2$ presented in the solar system solution vanishes, because now the FRW background is the right background. One can simply verify $C_0 = 0$ by the boundary condition that when $r \to \infty$, $\phi \to 0$, $\psi \to 0$ and $R \to R_{\mathrm{FRW}}$.

The Poisson equation (Eq. 8) is replaced by [8]

$$\nabla^2(\phi - \psi) = -\frac{8\pi G(\rho_m - \bar{\rho}_b)}{1 + f_R} \; . \qquad (14)$$

Here, ρ_m is the matter density of galaxies or clusters and $\bar{\rho}_b$ is the background matter density.

The gravitational lensing is governed by the combination $\phi - \psi$. So, given the same matter distribution, the gravitational lensing effect of a galaxy or a cluster in $f(R)$ gravity is identical to that in GR, except a change in the Newton's constant from G to $G/(1+f_R)$. For $f(R)$ to drive late time acceleration, $f_R > 0$ in general, thus the lensing signal will be smaller by a factor $f_R/(1+f_R) \simeq f_R$. However, for some $f(R)$ gravity models, $|f_R| \ll 1$ in galaxy and cluster environments, so the difference may not be observable.

However, galaxy rotation curve and intra-cluster gas pressure profile can be significantly different to that in GR. The acceleration of a test particle is $\dot{\mathbf{v}} = -\nabla\psi$. The gas pressure p is determined by $\nabla p = -\rho\nabla\psi$. The matter density decreases from $\sim 10^4\rho_c$ close to the center to $\sim 40\rho_c$ at virial radius. The corresponding variation in f_R is then comparable to variations in ϕ, ψ and thus $\phi + \psi = 0$ no longer holds. So in galaxy and cluster environments, ψ in $f(R)$ gravity does not follow the Poisson equation, as that in GR does. Whether the difference caused is observable is currently under investigation.

SUMMARY

To investigate $f(R)$ gravity in the solar system, galaxies and clusters, we derive the complete sets of the field equations which determine the two Newtonian potentials ϕ and ψ, under corresponding environments. We found $f(R)$ gravity models of cosmological interest behave indistinguishably to GR in the solar system.

We predict that gravitational lensing effect of quasi-static celestial objects such as galaxy and clusters in $f(R)$ gravity is virtually the same as in GR. However, galaxy rotation curve and cluster gas pressure profile differ intrinsically from that in GR. Thus observations of galaxies and cluster dynamics are promising to put useful constraints on $f(R)$ gravity.

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[18] The condition $|f_R| \ll 1$ is a sufficient condition for Eq. 7, however, not a necessary one. From Eq. 7, we get $(\phi+\psi)_{,ij}(1+f_R)+(\phi+\psi)f_{R,ij}+(\phi+\psi)_{,i}f_{R,j}+(\phi+\psi)_{,j}f_{R,i} = -f_{R,ij}$. Since $|\phi+\psi| \ll 1$, the last three terms in the left hand side is negligible comparing to the right hand side, we then obtain $(\phi+\psi)_{,ij}(1+f_R) \simeq -f_{R,ij}$, This is the Eq. 6 to begin with.